Bone Articulations as Systems of Poroelastic Bodies in Contact

J. L. NOWINSKI*
University of Delaware, Newark, Del.

Two-phase poroelastic material is taken as a model of a living bone, in the sense that the osseous tissue is considered as a perfectly elastic solid and the fluid substances filling the cavities as a viscous fluid. Using Heinrich-Desoyer equations, derived from the consolidation theory, equations for the normal displacements on the boundary of a half-infinite poroelastic solid subjected to a normal concentrated force are found. Theory identical with Hertz's theory for elastic bodies in contact is applied to the contact problem of poroelastic bodies. Two particular cases are analyzed in more detail: a) when the shape of the poroelastic bodies at the point of contact is spherical, whereas one of the bodies represent a spherical seat; and b) when the spherical seat becomes flat and its material perfectly rigid (a metallic prosthesis). Viscoelastic properties of the adopted bone model appear to be in full agreement with the experimental findings of Sedlin.

Nomenclature

$a_0,b_0;\ a_\infty,b_\infty$	= semiaxes of the contact ellipse at time $t = 0$ and ∞, respectively
A,B	= defined by Eq. (27)
f	= porosity
k_0	= coefficient defined after Eq. (4)
p	= excess fluid pressure
P	= external force
q,q_0 or q'	= continuous external force
R_1,R_2	= principal radii of curvature
r,z	= radial and axial coordinate, respectively
u,w	= displacement in radial and axial (or in x and z direction), respectively
v	= displacement in y direction
$w_0^i,w_\infty^i,\theta_0^i,\theta_\infty^i$, $(i = 1,2)$	= defined by Eq. (28)
$\bar{u},u^*$	= Laplace and Hankel transform of u, respectively
x,y,z	= Cartesian orthogonal coordinates
x',y',z'	= coordinates of load application point
α	= defined by Eq. (25) as so-called drift
α,ν,λ,μ	= material constants
γ_w	= specific weight of fluid
δ	= defined by Eq. (34)
ϵ	= volume dilatation
κ	= coefficient of permeability
$\sigma_{rr},\sigma_{\phi\phi},\sigma_{zz},\sigma_{zr}$	= stress components
ξ,η	= Laplace and Hankel transform parameters, respectively

Introduction

DURING the period after World War II, an extensive literature on behavior and mechanical properties of human bone was given to the world, as a consequence of an unprecedented development of medical physics and biomedical engineering in general. This happening should be considered quite natural. In fact, our knowledge of the human skeleton as a load carrying system, so vital for mankind, was and continues to be in no more than a prenatal state as compared, for instance, with our comprehension of mechanical behavior of technological structures.

A summary of studies on the human bone, up to 1957, was given by Evans in his classical book.[1] Some recent contributions were briefly reviewed, on the broad basis of biomechanics in general, by Fung.[2] The entire existing literature was analyzed in detail by Kraus in his broad synthetic review,[3] in 1968. Despite the profusion of work, the attempts at a quantitative mathematical analysis of the states of stress and deformation in the human bone are very scarce. Actually, as far as it is known (cf. also Ref. 3), such studies were conducted only by Wertheim[4] and Rauber[5] about a hundred years ago; then by Koch,[6] fifty years ago, and more recently by Marique[7] in 1945. All the work was concerned mostly with the femur, that is the thigh-bone, which was treated as an elastic beam. Koch's is the most extensive analysis, but in his paper, besides other inaccuracies, the nonhomogeneity of the cross-sections of the bone, that reveal both cancellous (spongy) and compact components, was not properly accounted for. It should be noted that in all these endeavors a rather elementary, and occasionally naive, stress analysis was employed, based on simplified rules of the Strength of Materials.

The present paper represents an attempt at a more scrupulous quantitative analysis of the mechanical behavior of bones, in particular of the bone articulations (joints). The analysis is modelled on the Terzaghi-Biot (see Refs. 8–11) and the Terzaghi-Heinrich-Desoyer[12] theory of consolidation. It belongs to future investigations to determine whether such an approach fully describes the actual behavior of living bones.

As is well known, the osseous tissue resembles the latticework, in the sense that it contains innumerable cavities in a solid skeleton. The cavities (pores) are filled with bone marrow and various fluids such as blood, synovial fluid etc. The microscopic structure also shows that the skeleton itself is perforated by tiny passages which are so numerous that in one cubic millimeter of bony tissue there are millions of these passages (canaliculi, lacunae, Haversian and Volkmann canals, etc.), cf. Ref. 3, p. 172. As a consequence, the differences between two main types of osseous tissue, seemingly distinct, that is the spongy and the compact bone, are rather relative and consist in the proportion of the volume of the pores to the volume of the solid matter (skeleton), in a unit volume of the bulk material.

As most biological materials, a real bone is nonhomogeneous and anisotropic. Furthermore, its solid phase is, most likely, governed by nonlinear constitutive equations, and its liquid phase behaves like a non-Newtonian fluid.

To facilitate the calculations, in the present study, we make some natural simplifications. We first assume that the volume concentration of pores is uniform, so that the bulk material may be regarded accordingly as quasi-homogeneous

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* H. Fletcher Brown, Professor of Mechanical and Aerospace Engineering.

and quasi-isotropic. Of course, these restrictions may be removed at a cost of computational labor. As a result of these assumptions the porosity of the material, defined as the percentage of the pore volume in the unit bulk volume, (or, what comes to the same, as the pore area in the unit cross-sectional bulk area, e.g., Ref. 13, p. 901) remains constant.

Furthermore, we assume that in the two-phase solid-fluid system considered, the solid skeleton is linearly and perfectly elastic and undergoing small deformations. Its mechanical behavior is, therefore, governed by Hooke's law. The liquid phase is considered Newtonian viscous, the pores (in view of what was said earlier) interconnected, and the flow of fluid, produced by the deformation of the bone, governed by Darcy's law. With regard to the internal forces, it is postulated that the stress in the bulk material is smoothly divided between and uniformly distributed over the solid phase, as the stress σ_{ij}, and over the liquid phase, as the excess pressure p (both per unit area of the corresponding constituents). This assumption is the most vulnerable, since the presence of cavities in the spongy bone certainly contributes to the local concentrations of stress, so that, in fact, the mechanical picture of the phenomenon is extremely complicated. As a consolation, one may accept Currey's opinion[14] (see also Ref. 3, p. 181), that the discontinuities in the bone are generally arranged in such a fashion that they create minimum stress concentrations. Moreover, the propagation of eventual cracks is rather prevented, much in the same way as in the Bowie's crack model (holes drilled at the tips of the crack).

The bone articulations, which are the topic of the present study were analyzed qualitatively from the point of view of the viscoelastic consolidation model by Zarek and Edwards.[16]† More accurately, their speculations concern the articular cartilage, that is the thin covering over the osseous faces which meet at a joint.

In view of the adopted model, in the present paper we represent the skeletal joint as a system of two poroelastic bodies in contact, pressed together by an external load. The well known theory of local stresses between bodies in contact derived by Hertz for elastic bodies is easily extended to the poroelastic case. The surface of contact created by the external load being small as compared with the dimensions of the bodies, makes it legitimate to treat the local deformation as the deformation of a semi-infinite medium with a plane boundary. In the poroelastic case, the associated, so-called Boussinesq problem, was solved by several authors; namely by Paria,[17] McNamee and Gibson,[18] and Heinrich and Desoyer.[12] In what follows we apply the equations derived by the last two authors, since they seem the most explicit and suitable for applications in mind.

Two particular cases are treated in more detail: a) when the poroelastic bodies in contact have spherical shape at the point of contact, whereas one body forms a spherical seat for the other; and b) when the spherical seat degenerates into a plane and its material becomes perfectly rigid (a metallic prosthesis). The analysis concerns the initial moment after the application of load ($t = 0$) and at the asymptotic stage ($t = \infty$).

It is found that in both cases the initial contact area increases with the time up to 30%, reflecting the ability of living organisms to diminish the average (and the maximum) pressure at the sites of contact. Owing to this property, the final maximum pressure exceeds the average initial pressure by no more than 15%. With regard to a metallic element in contact with a bone, the contact area in this case is almost 60% less than the corresponding area in the bone-to-bone contact. In the same proportion, that is by 60%, increases the average and the peak stress, both at the initial moment and at the time $t = \infty$.

Viscoelastic properties of the adopted bone model show the pattern followed by the standard linear three-parameter viscoelastic system (elastic spring and Kelvin-Voigt element in series). Under a sudden application of constant load, instantaneous deformation originates which, as the time goes by, increases asymptotically to a final bounded value. This increase, for the value of the material characteristic ν taken equal to the experimentally determined value of Poisson's ratio for the bovine femur, amounts to 14%. The phenomenon of creep occuring under the action of constant load, as well as our earlier findings of stress relaxation under constant strain,[15] display a full agreement with the model proposed by Sedlin on the basis of his experimental observations.[20,21] In fact, according to Sedlin, under loads well below the fracture load, the actual behavior of bones is well qualitatively characterized by the standard linear viscoelastic model. This is just the model suggested by the consolidation theory.

1. General Equations

As mentioned earlier, in the present analysis we employ the results of the solution of Heinrich and Desoyer to the Boussinesq problem for a semi-infinite poroelastic medium.[12] For details, we refer the reader to the original paper. However, to make our analysis self-contained and for future reference, we briefly recapitulate the main points of the solution.

Let us consider a semi-infinite body of poroelastic material referred to a cylindrical coordinate system r, ϕ, z, with the plane $r\phi$ coinciding with the bounding plane and the z axis pointing towards the interior of the body. Assume, for the time being, that the deformation of the body is axially symmetric with z axis as symmetry axis. Assume also that the pores at the boundary $z = 0$ are unsealed (which seems to be a realistic assumption for a living bone). Denote the radial and the axial displacement components of the solid phase by $u = u(r,z;t)$ and $w = w(r,z;t)$, respectively, (t = time). Denote, furthermore, by $p = p(r,z;t)$ the fluid excess pressure in the pores (that is the magnitude of pressure over the environmental pressure), by f the ratio of the area covered by the liquid to the total area of elementary cross-sections, and by ϵ the volume dilatation

$$\epsilon = (1/r)(\partial/\partial r)(ur) + \partial w/\partial z \tag{1}$$

It is assumed that the deformation of the skeleton is brought about by the change of the pore volume only (the dilatation of the solid phase itself being disregarded). The fundamental equations of the problem then become[12]

$$\begin{aligned} [\nabla^2 - (1/r^2)]u + [1/(1-2\nu)]\partial\epsilon/\partial r &= 2\alpha(1+\nu)f\partial p/\partial r \\ \nabla^2 w + [1/(1-2\nu)]\partial\epsilon/\partial z &= 2\alpha(1+\nu)f\partial p/\partial z \end{aligned} \tag{2}$$

$$\nabla^2\epsilon = \kappa\partial\epsilon/\partial t \tag{3}$$

$$\begin{aligned} \nabla^2 p &= (\gamma_w/k_0)\partial\epsilon/\partial t \\ \nabla^2 &= \partial^2/\partial r^2 + (1/r)(\partial/\partial r) + \partial^2/\partial z^2 \end{aligned} \tag{4}$$

In the foregoing equations α and ν denote material constants characterizing the deformation of the skeleton, κ is the coefficient of permeability of the material, γ_w is the specific weight of the fluid phase, and $k_0 = [\alpha(1+\nu)(1-2\nu)]\cdot f\gamma_w/[(1-\nu)\kappa]$. Clearly, Eqs. (2) represent the equations of equilibrium of the bulk material, and Eq. (4) phrases Darcy's law of flow of the fluid phase.

By the symmetry of the deformation the stress components $\sigma_{r\phi}$ and $\sigma_{z\phi}$ in the skeleton vanish, and the remaining stress

† This paper came to the attention of the writer after completion of the present and an earlier paper.[15]

components become

$$\begin{aligned}\sigma_{rr} &= 1/\alpha(1+\nu)\{(\partial u/\partial r)\nu/(1-2\nu)\epsilon\}\\ \sigma_{\phi\phi} &= 1/\alpha(1+\nu)\{(u/r)+\nu/(1-2\nu)\epsilon\}\\ \sigma_{zz} &= 1/(1+\nu)\{(\partial w/\partial z)+\nu/(1-2\nu)\epsilon\}\\ \sigma_{zr} &= 1/2\alpha(1+\nu)[(\partial u/\partial z)+\partial w/\partial r]\end{aligned} \tag{5}$$

The system of coupled Eqs. (2–4) is solved by first disposing of the time variable by means of Laplace transform, and then by relegating the r variable by means of the zero-order Hankel transform. To this purpose it is first necessary to define the initial and boundary conditions. A realistic choice is that at the time $t = 0$ a normal axially symmetric load $q = q(r,t)$ is suddenly applied at the plane $z = 0$ and thereafter kept constant. Except for the tractions $q(r,t)$, the plane $z = 0$ should be free from load and since, by hypothesis, it is perfectly permeable for the fluid phase, at all times $t > 0$, we have

$$\left.\begin{aligned}\sigma_{rz} &= 0\\ \sigma_{zz} &= -q(r)\\ p &= 0\end{aligned}\right\} \text{ on } z = 0 \text{ at } t > 0 \tag{6}$$

Clearly, we treat the problem as quasi-static.

The solution to the problem proceeds now in the following steps: a) first, the dilatation ϵ is determined from Eq. (3) with accuracy to an unknown integration function; b) then p is found from Eq. (4) involving another unknown integration function; c) finally, the displacement functions u and v are determined from Eqs. (2) involving two unknown functions. If ξ denotes the Laplace transform parameter, and η the Hankel transform parameter, then the four unknown integration functions become functions of ξ and η; they are determined from the conditions of Eq. (6) and the relation of Eq. (1).

Denote by a bar the Laplace transform and by an asterisk the Hankel transform of order zero. Then a Laplace-Hankel transform of the load $q(r,t)$ becomes

$$\bar{q}^*(\xi,\eta) = \int_0^\infty \eta \int_0^\infty e^{-\xi t} q(r,t)dtJ_0(r\eta)d\eta \tag{7}$$

(J_0 = Bessel function of the first kind and zero order). A longer computation furnishes now the Laplace transforms of u and w in the following form:

$$\begin{aligned}\bar{u} = \alpha(1+\nu)\int_0^\infty \times\\ &\frac{[(1-\nu)\kappa\xi z - (1-2\nu)(\eta^2+\kappa\xi)^{1/2}]e^{-\eta z} +}{(1-2\nu)\eta[\eta - (\eta^2+\kappa\xi)^{1/2}] + (1-\nu)\kappa\xi}\\ &\frac{(1-2\nu)\eta e^{-(\eta^2+\kappa\xi z)^{1/2}}}{(1-2\nu)\eta[\eta-(\eta^2+\kappa\xi)^{1/2}]+(1-\nu)\kappa\xi} \times\\ &\bar{q}^*(\xi,\eta)\eta J_1(r\eta)d\eta\end{aligned} \tag{8}$$

$$\begin{aligned}\bar{w} = \alpha(1+\nu)\int_0^\infty \times\\ &\frac{[(1-\nu)\kappa\xi(1+\eta z) -}{(1-2\nu)\eta(\eta-\eta^2+\kappa\xi)+(1-\nu)\kappa\xi}\\ &\frac{(1-2\nu)\eta(\eta^2+\kappa\xi)^{1/2}]e^{-\eta z} +}{(1-2\nu)\eta(\eta-\eta^2+\kappa\xi)+(1-\nu)\kappa\xi}\\ &\frac{(1-2\nu)\eta(\eta^2+\kappa\xi)^{1/2}e^{-(\eta^2+\kappa\xi z)^{1/2}}}{(1-2\nu)\eta(\eta-\eta^2+\kappa\xi)+(1-\nu)\kappa\xi} \times\\ &\bar{q}^*(\xi,\eta)J_0(r\eta)d\eta\end{aligned}$$

A retransformation from the Laplace subspace requires an explicit definition of the surface load q. For our future purposes it is expedient to consider the load as a concentrated force P applied at the origin of coordinates. To solve this problem, it is customary first to assume that at all times $t > 0$

$$q(r,t) = q_0 \text{ for } r < a \text{ and } q(r,t) = 0 \text{ for } r > a \tag{9}$$

where q is a constant load and a the radius of a circle with center at the origin of coordinates. The Laplace-Hankel transform gives now

$$\bar{q}^*(\xi,\eta) = (q_0 a/\xi\eta)J_1(a\eta) \tag{10}$$

and upon passing to the limit with $a \to 0$ and the value $q_0 a^2\pi \equiv P$ kept constant, we finally obtain

$$\bar{q}^*(\xi,\eta) = P/2\pi\xi \tag{11}$$

Clearly, P represents the desired concentrated force acting at the origin. With this load in mind a lengthy calculation leads to the displacement components in the following form:‡

$$\begin{aligned}u = \alpha(1+\nu)\frac{P}{2\pi}\int_0^\infty \Big\{e^{-\eta z}\Big[\Big[(1-2\nu)\frac{\eta^2 t}{\kappa} \times\\ \Big[\mathrm{Erfc}\left(\frac{\eta^2 t}{\kappa}\right)^{1/2} + \mathrm{erfc}[(\eta z^2/4t)^{1/2} - (\eta^2 t/\kappa)^{1/2} - 2]\Big] +\\ 2(1-\nu)\eta z - \tfrac{1}{2}(1-4\nu^2) - \tfrac{1}{4}(1-2\nu)(2\eta z + 1 - 2\nu) \times\\ \mathrm{erfc}[(\kappa z^2/4t)^{1/2} - (\eta^2 t/\kappa)] - [(1-\nu)\eta z - \tfrac{1}{2}(1-2\nu^2)] \times\\ \mathrm{erfc}[(\eta^2 t/\kappa)^{1/2} + \nu(\eta z - \nu)e^{[1-2\nu/(1-\nu)^2]\eta^2 t/\kappa}]\ \mathrm{erfc} \times\\ [(\nu/1-\nu)(\eta^2 t/\kappa)^{1/2}] - (1 - 2\nu/\pi^{1/2})(\eta^2 t/\kappa)e^{-\eta^2 t/\kappa}/\pi^{1/2}\Big]\Big] +\\ \tfrac{1}{4}e^{\eta z}\mathrm{erfc}[(\kappa z^2/4t)^{1/2} + (\eta^2 t/\kappa)^{1/2}] - \nu(1-\nu)\exp \times\\ \Big[-\frac{1-2\nu}{(1-\nu)^2}\eta^2 t/\kappa) + \frac{\nu}{1-\nu}\eta z\Big]\mathrm{erfc}\ [(\kappa z^2/4t)^{1/2} +\\ \Big[\kappa\nu\left(\frac{\eta^2 t}{\kappa}\right)^{1/2}\Big/(1-\nu)\Big] + (1-2\nu/\pi^{1/2})(\eta t^2/\kappa)\exp -\\ [(\kappa z^2/4t) + (\eta^2 t/\kappa)]\Big\}J_1(r\eta)d\eta\end{aligned} \tag{12}$$

$$\begin{aligned}w = \alpha(1+\nu)\frac{P}{2\pi}\int_0^\infty \Big\{e^{-\eta z}\Big[\Big[(1-2\nu)\frac{\eta^2 t}{\kappa} +\\ \Big[\mathrm{erfc}\left(\frac{\eta^2 t}{\kappa}\right)^{1/2} + \mathrm{erfc}\left(\frac{\kappa z^2}{4t}\right)^{1/2} - \left(\frac{\eta^2 t}{\kappa}\right)^{1/2} - 2\Big] +\\ \nu)\eta z + 2(1 - \tfrac{1}{2}(3 - 4\nu + 4\nu^2) - \tfrac{1}{4}[2(1-2\nu)\eta z - (1 -\\ 4\nu^2)]\ \mathrm{erfc} \times [(\kappa z^2/4t)^{1/2} - (\eta^2 t/\kappa)] - [(1-\nu)\eta z + \tfrac{1}{2}(1 - 2\nu +\\ 2\nu^2)] \times \mathrm{erfc}\ (\eta^2 t/\kappa)^{1/2} + \nu(1 + \eta z - \nu)\exp\left[\frac{2\nu-1}{(1-\nu)^2}\frac{\eta^2 t}{\kappa}\right] \times\\ \mathrm{erfc}[(\nu/1-\nu)(\eta^2 t/\kappa)^{1/2}] - (1 - 2\nu/\pi^{1/2})(\eta^2 t/\kappa)^{1/2}e^{-\eta^2 t/\kappa}\Big]\Big] -\\ \tfrac{1}{4}e^{\eta z}\ \mathrm{erfc}[(\kappa z^2/4t)^{1/2} + (\eta^2 t/\kappa)^{1/2}] + \nu^2\exp\Big[\frac{\nu}{1-\nu} \times\\ \eta z - \frac{1-2\nu}{(1-\nu)^2}\frac{\eta^2 t}{\kappa}\Big]\mathrm{erfc}[(z\kappa^2/4t)^{1/2} + (\nu/1-\nu)(\eta^2 t/\kappa)^{1/2}] +\\ (1 - 2\nu/\pi^{1/2})(\eta^2 t/\kappa)^{1/2}\exp[(\kappa z^2/4t) + (\eta^2 t/\kappa)]\Big\}\ J_0(r\eta)d\eta\end{aligned} \tag{13}$$

In the subsequent analysis, we shall be mostly concerned with the vertical displacement $w(r,z;t)$. However, using the clumsy form in Eq. (13) for w or the form in Eq. (12) for u would rather obscure than clarify the discussion. Instead, we content ourselves with the analysis of two limit cases, associated with the initial moment after the application of load, $t = +0$, and the final state $t = \infty$. In these cases

‡ Except for the loading terms and the partially altered notation, Eqs. (12) and (13) are identified as Eqs. (157) and (158) in Ref. 12.

Eq. (13) reduces to one of the following equations:

$$w_{\substack{t=+0\\ z>0}} = \alpha(1+\nu)\frac{P}{2\pi}\int_0^\infty (1+\eta z)e^{-\eta z}J_0(r\eta)d\eta \quad (14)$$

and

$$w_{\substack{t=\infty\\ z>0}} = \alpha(1+\nu)\frac{P}{2\pi}\int_0^\infty [\eta z + 2(1-\nu)]e^{-\eta z}J_0(rn)d\eta \quad (15)$$

Similarly for the radial displacement component we obtain

$$u_{\substack{t=+0\\ z>0}} = \alpha(1+\nu)\,P/2\pi\int^\infty z e^{-\eta z}J_1(r\eta)d\eta \quad (16)$$

$$u_{\substack{t=\infty\\ z>0}} = \alpha(1+\nu)\frac{P}{2\pi}\int^\infty [\eta z - (1-2\nu)]e^{-z\eta}J_1(r\eta)d\eta \quad (17)$$

To avoid serious mathematical difficulties it is expedient, at this junction, to transfer from cylindrical to the Cartesian rectangular coordinates, x, y, and z (with xy plane coinciding with the boundary of the half-space). This change enables us to make later a direct use of Hertz's mastery of an intricate mathematical problem. With the notation $\rho = (r+z)^{1/2} = (x^2+y^2+z^2)^{1/2}$ we first find the closed form of the following improper integrals:

$$\int_0^\infty e^{-\eta z}J_0(r\eta)d\eta = \frac{1}{\rho}\int_0^\infty e^{-\eta z}J_0(r\eta)d\eta = \frac{z}{\rho^3} \quad (18)$$

$$\int_0^\infty e^{-\eta z}J_1(r\eta)d\eta = \frac{1}{r}\left(1-\frac{z}{\rho}\right)\int_0^\infty \eta e^{-zy}J_1(r\eta)d\eta = \frac{r}{\rho^3} \quad (19)$$

Thereafter, we denote by u, v, and w the displacement components along the x, y, and z axis, respectively.§

With Eq. (18) and Eq. (19) in mind we easily find at $t = 0$

$$\begin{aligned} u_{\substack{t=0\\ z>0}} &= (P/4\pi\mu)xz/\rho^3 \\ v_{\substack{t=0\\ z>0}} &= (P/4\pi\mu)yz/\rho^3 \\ w_{\substack{t=0\\ z>0}} &= (P/4\pi\mu)(z^2/\rho^3) + (P/4\pi\mu)1/\rho \end{aligned} \quad (20)$$

similarly, at $t = \infty$

$$\begin{aligned} u_{\substack{t=\infty\\ z>0}} &= (P/4\pi\mu)(xz/\rho^3) - [P/4\pi(\lambda+\mu)]\,x/\rho(z+\rho) \\ v_{\substack{t=\infty\\ z>0}} &= (P/4\pi\mu)(yz/\rho^3) - [P/4\pi(\lambda+\mu)]y/\rho(z+\rho) \\ w_{\substack{t=\infty\\ z>0}} &= (P/4\pi\mu)(z^2/\rho^3) + \\ &\qquad [P(\lambda+2\mu)/4\pi\mu\ (\lambda+\mu)]1/\rho \end{aligned} \quad (21)$$

with the new notation $\mu = 1/2\alpha(1+\nu)$ and $\lambda = \nu/\alpha(1+\nu)(1-2\nu)$. Eqs. (21), associated with the time $t = \infty$, represent exactly the Boussinesq formulas for the purely elastic case, (Ref. 19, p. 191). Thus, the process of deformation follows the following pattern. At the moment of application of load, instantaneous deformation originates, represented by Eqs. (20). With the time going by, the system starts to creep, in the sense that the deformations change gradually and converge asymptotically towards their final values represented by Eqs. (21). Specifically, the vertical displacement component w increases with time, and two other components decrease with time and even, possibly, change their signs. Both these phenomena, that is the instantaneous

§ No confusion should arise from the present notation u, used earlier for the radial displacement component.

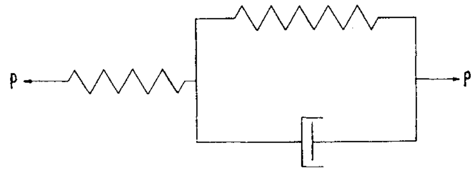

Fig. 1 Standard linear model.

response and the creep of bounded magnitude, are characteristic of the so-called standard linear three-parameter viscoelastic solid, composed of an elastic spring and a Kelvin-Voigt element in series, Fig. 1. This model agrees with the model proposed by Sedlin, on the basis of his experimental observations on bones, as characterizing well qualitatively the behavior of bones under (fixed) load well below the fracture load (Refs. 20, 21, cit. after Ref. 3, p. 195). On the other hand, the bare Kelvin-Voigt model proposed for the bony materials in Ref. 16 seems to be too particular. Let us note, that the quantitative analysis of the bony materials subjected to fixed strain, based on the Biot consolidation theory, revealed the phenomenon of stress relaxation progressing with time.[15] This would rather confirm the correctness of the standard model than of the simple Kelvin-Voigt model in which relaxation process is absent.

If these findings are true the constitutive equation of the bony materials (in the simple one-dimensional case) may be defined by

$$n(\partial\sigma/\partial t) + \sigma = Hn(\partial e/\partial t) + Ee \quad (22)$$

where H and E are often called the instantaneous and extended elastic moduli, and n is the time of relaxation (cf. also Ref. 3). Accepting the constitutive Eq. (22) it is feasible, of course, to predict realistically the mechanical behavior of bony structural elements in any circumstances by applying the present general theory.

Returning to our main problem, one first shifts the force—by the well known procedure—to a generic point x', y', o, and with notation $q(x',y') = P/dx'dy'$ and $\rho = [(x-x')^2 + (y-y')^2 + z^2]^{1/2}$, one obtains for the displacement components generated by an arbitrarily distributed continuous load $q(x',y')$

$$\begin{aligned} u_{t=0} &= z/4\pi\mu\int[(x-x')q'/\rho^3]dx'dy' \\ v_{t=0} &= z/4\pi\mu\int[(y-y')q'/\rho^3]dx'dy' \\ w_{t=0} &= z^2/4\pi\mu\int(q'/\rho^3)dx'dy' + 1/4\pi\mu\int(q'/\rho)dx'dy' \end{aligned} \quad (23)$$

$$\begin{aligned} u_{t=\infty} &= z/4\pi\mu\int[(x-x')q'/\rho^3]dx'dy' - \\ &\qquad [1/4\pi(\lambda+\mu)]\int[(x-x')q'/\rho(z+\rho)]dx'dy' \\ v_{t=\infty} &= z/4\rho\mu\int[(y-y')q'/\rho^3]dx'dy' - \\ &\qquad [4/4\pi(\lambda+\mu)]\int[(y-y')q'/\rho(z+\rho)]dx'dy' \\ w_{t=\infty} &= z^2/4\pi\mu\int(q'/\rho^3)dx'dy' + \\ &\qquad [(\lambda+2\mu)/4\pi\mu(\lambda+\mu)]\int(q'/\rho)dx'dy' \end{aligned} \quad (24)$$

Here, of course, the integrals are spanned over the loaded area. We are now ready to turn to the final problem, that is to the contact problem itself; it is analyzed in the next section.

2. Poroelastic Bodies in Contact

Let us consider two poroelastic bodies, I and II, touching each other at a point and pressed together along the common normal at the point of contact 0. We take 0 as the origin of two coordinates systems x, y, z_1 and x, y, z_2 with the coordinate plane xy as the common tangent plane at 0 (clearly, perpendicular to the common normal at 0); z_1 and z_1 are directed towards the interior of the bodies I and II, respectively, Fig. 2. We assume that 0 is a regular point of the

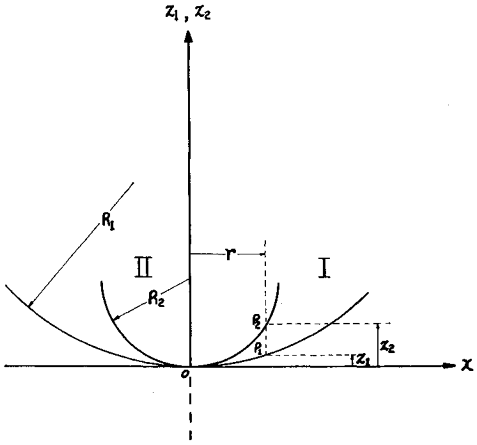

Fig. 2 Bodies in contact.

surfaces $z_1 = z_1(x,y)$ and $z_2 = z_2(x,y)$. To simplify the calculations we assume that the planes xz_1, yz_1, xz_2, and yz_2 are planes of principal curvature with radii of curvature R_1, R_1', and R_2, R_2', respectively.¶ To simulate the ordinary situation at joints, we take that R_1 and R_1' are negative, so that body I represents a concave seat. The kinematical part of Hertz's theory is valid for all deformable bodies, irrespective of their material, provided the deformation is infinitesimal.** This part is therefore applicable to poroelastic bodies as well. When the bodies are pressed together the sum of the displacements of particles on any common normal such as P_1P_2, is equal to the sum of displacements of the particles in contact at 0, (this may be called the drift α) plus the sum of the vertical displacements of the particles at P_1 and P_2, w_1, and w_2. In order for P_1 and P_2 to coalesce on the surface of contact produced by the compressive load around the origin 0, the condition

$$\alpha - (w_1 + w_2) = z_2 - z_1 \tag{25}$$

must be satisfied, where

$$z_2 - z_1 = Ax^2 + By^2 \tag{26}$$

and

$$A = \tfrac{1}{2}[(1/R_2) - (1/R_1)],\ B = \tfrac{1}{2}[(1/R_2') - (1/R_1')] \tag{27}$$

Clearly, the coefficients A and B are known if the shape of the bodies in contact is prescribed. It is shown in Hertz's theory that the surface of contact is generally a small ellipse, whose axes—in the present case—coincide with the axes of the ellipses in Eq. (26), that is the ellipses formed by points at the same distance P_1P_2 from each other before the deformation. It follows, that the axes of the surface of contact, call them 2a and 2b, are located on the x and y axis, respectively.

If a and b are very small as compared with the radii of curvature, the two bodies in contact may be substituted by two semispaces in contact. It is, therefore, legitimate to apply the results of the preceding section, considering the forces arising on the surface of contact as the forces acting on each of the poroelastic semispaces. From Eqs. $(23)_3$ and $(24)_3$ we see that on the surface of contact

$$w_0^1(x,y) = \theta_0^1\phi_0(x,y),\ w_\infty^1(x,y) = \theta_\infty^1\phi_0(x,y)$$
$$w_0^2(x,y) = \theta_0^2\phi_0(x,y),\ w_\infty^2(x,y) = \theta_\infty^2\phi_0(x,y) \tag{28}$$

where, e.g., w_0^1 denotes the vertical displacement of the body I at the initial moment, and θ_0^1 the associated coefficient $1/4\pi\mu_1$. Similarly, w_∞^2 denotes the asymptotic value of the displacement of the body II and $\theta_\infty^2 = (\lambda_2 + 2\mu_2)/4\pi\mu_2(\lambda_2 + \mu_2)$, with subscripts at the material constants indicating body II. Clearly, $\phi_0(x,y) = [(q'/\rho)dx'dy']_{z=0}$ is the ordinary Newtonian potential of a simple layer of density $q'(x',y')$ spread over the surface of contact. The problem is now to find the distribution function $q'(x',y')$ that is the interaction of the bodies across the surface of contact, from the condition in Eq. (25) in which Eqs. (26) and (27) are substituted. In this fashion we obtain at time $t = 0$

$$(\theta_0^1 + \theta_0^2)\phi_0(x,y) = \alpha - (Ax^2 + By^2) \tag{29}$$

and at the time $t = \infty$

$$(\theta_\infty^1 + \theta_\infty^2)\phi_0(x,y) = \alpha - (Ax^2 + By^2) \tag{30}$$

The difference between those two integral equations lies, of course, solely in the values of the material coefficients. The corresponding rather intricate problem was solved by Hertz, who found that the density $q(x',y')$ is obtained by considering ϕ_0 as the potential of an ellipsoid erected over the contact surface, if its diameter c along the z axis tends to zero, while the mass of the ellipsoid remains constant.

Final results specified to the case under discussion are as follows. The semiaxis of the elliptical contact surface, at time $t = 0$ and $t = \infty$, respectively, are (using approximate formulas):

$$a_0 = m\left[\frac{3\pi P(\theta_0^1 + \theta_0^2)}{4(A+B)}\right]^{1/3} \quad a_\infty = m \times \left[\frac{3\pi P(\theta_\infty^1 + \theta_\infty^2)}{4(A+B)}\right]^{1/3}$$
$$b_0 = n\left[\frac{3\pi P(\theta_0^1 + \theta_0^2)}{4(A+B)}\right]^{1/3} \quad b_\infty = n \times \left[\frac{3\pi P(\theta_\infty^1 + \theta_\infty^2)}{4(A+B)}\right]^{1/3} \tag{31}$$

where P denotes the total compressive force and m and n are tabulated for various values of the angle τ, defined by the equation

$$\cos\tau = (B - A)/(B + A) \tag{32}$$

Let us discuss two particular cases: a) Suppose, first, that the bodies in contact have spherical shape in a sufficient large neighborhood of the point of contact. Then

$$1/R_1' = 1/R_1,\ 1/R_2' = 1/R_2 \tag{33}$$

so that $A = B = 1/2[(1/R_2) - (1/R_1)]$ and $\tau = \pi/2$. From the table in Ref. 22, p. 176 we find, that $m = n = 1$, so that by Eq. (31) the contact ellipse degenerates into a circle of radius $a_0 = b_0$ and $a_\infty = b_\infty$, respectively. Suppose now that the material properties of the bony members that meet at the point are identical so that $\theta_0^1 = \theta_0^2 = \theta_0$, and $\theta_\infty^1 = \theta_\infty^2 = \theta_\infty$. The ratio δ of the radii of the contact circle, at the time $t = \infty$ and $t = 0$, becomes

$$\delta = a_\infty/a_0 = (\theta_\infty/\theta_0)^{1/2} = [(\lambda + 2\mu)/(\lambda + \mu)]^{1/3} \tag{34}$$

To get a rough estimate of the magnitude of δ, assume that the material constant ν in Eq. (34) is equal to Poisson's ratio of the wet bovine femur (according to Ref. 23, Table 1), that is $\nu = 0.26$. Then

$$\delta = [2(1 - \nu)]^{1/3} = 1.14 \tag{35}$$

It follows, that the contact area increases with time up to 30%, and at the same ratio decreases the average load per unit area. Thus, a living organism, subjected to an additional load, possesses a natural ability to enlarge the area of arising contact, and to diminish the average stress at sites of contact. As mentioned earlier, the interacting pressure

¶ It is well known that this simplification is avoided in the general theory of bodies in contact.[19]

** There are some restrictions on the exactness of the results, if the bodies are of different materials, see Ref. 19, footnote on p. 195.

$q(x',y')$ is equivalent to the density of ellipsoidal mass distribution over the contact area, if the peak of the distribution c goes to zero and the total mass of the distribution remains constant. In the present case, for a spherical pressure distribution, we have at some point x', y', of the contact area,

$$q(x',y') = 2 \lim_{c \to 0}(c\rho)\{1 - [(x')^2 + (y')^2/a^2]\} \quad (36)$$

where ρ is the volumetric mass density of the distribution, and the total mass of ellipsoid may be taken equal to the load P

$$P = \tfrac{4}{3}\pi a^2 c\rho \quad (37)$$

Hence

$$q(x',y') = 3P/2\pi a^2\{1 - [(x')^2 + (y')^2/a]\} \quad (38)$$

and the maximum pressure q_0 (at $x' = y' = 0$) is equal to

$$q_0 = 3P/2\pi a^2 \quad (39)$$

Since, in view of Eq. (39) the maximum pressure is in inverse proportion to the area of contact, its value diminishes with time by 30%, as found earlier for the average pressure. Thus, the final maximum pressure exceeds the average initial pressure by no more than 15%. and b) Assume now that the bony element remains in contact with a metallic prosthesis. Let the bone, called body II, be spherically shaped at the point of contact, and the prosthesis, call it body I, considered perfectly rigid ($\theta^1 \equiv 0$) and flat ($R_1 = R_1' = \infty$). Then $A = B = \frac{1}{2}R$ and from Eq. (31)

$$a_0 = (3\pi PR\theta_0/4)^{1/3},\ a_\infty = (3\pi PR\theta_\infty/4)^{1/3} \quad (40)$$

It is seen that, in the present case, the coefficient of magnification of the contact area δ remains the same as in the preceding case, that is defined by Eq. (35). However, the area of contact diminishes as compared with the case in which the metallic prosthesis is substituted by a flat bone. In fact, in the latter case $\theta^1 \neq 0$ and for $\theta^1 = \theta^2 = \theta$, Eqs. (31) give

$$a_0 = (3\pi PR\theta_0/2)^{1/3},\ a_\infty = (3\pi PR\theta_\infty/2)^{1/3} \quad (41)$$

Thus the area of contact of a metallic prosthesis with a bone is by almost 60% less than the corresponding area in the bone-to-bone case. In the same proportion, that is by 60%, increases the average and the maximum stress, both at the initial moment and at time $t = \infty$. The contact of two bony members is, therefore, better serviceable than the contact of the bone with a prosthesis.

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